Analogy in Physical Action

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Analog methods in engineering were developed in the 1800's, right about the time when Maxwell introduced the idea that electrical actions were dynamic. In 1914 Max Jakob developed the similarity between thermal and magnetic fields and used analogy to study and test insulation materials (*Heat Transfer*, by Max Jakob, 1949). Concepts in electrical networks were extended to mechanical and acoustic networks.

The basic concept behind this pattern of analogy is the relationship between impedance and flow as factors in pressure. Pressure (or force, or electromotive force, etc.) is considered an "across" variable since it is measured across two points in a system, while flow (or velocity, or current, etc) is considered a "through" variable since it is a measure of flux through a conductor.

For a simple and familiar example, consider the electrical formula: E=Li'+Ri+(1/C)q and compare to the mechanical formula: F=Ma+Dv+Kx, where i=q/t and v=x/t, etc. The triple [M,D,K] represents mechanical impedance, which includes both resistance (D, friction) and reactance (M, inertial mass; and K, spring constant or compliance). Likewise the triple [L,R,1/C] represents electrical impedance, which includes both resistance (R) and reactance (L, inductance; and 1/C, 1/capacitance).

The table of analogues that are presented here is like a "Rosetta Stone for physics" – the constitutive equations for numerous energetic domains have a parallel structure, known as mathematical isomorphism. The constitutive equations are given as ordinary second-order linear differential equations, and while it is quite possible to extend the analogy beyond these simplified forms, they do have validity as an approximation, and just the fact that energy transfer shows this parallel behavior is worth consideration and analysis. Possible extensions will be considered presently.

In the following table, the rightmost three columns are impedance: inductive reactance, resistance, capacitive reactance. Each column is the time derivative of the previous one, so the dimensionality is the same except the power of time is reduced by 1. Column 5, Resistance, is the real component of impedance, with columns 4 and 6 representing Reactance (inductive and capacitive), modeled as the complex part of Impedance. The third column is the flow variable (velocity, current), and the second column is the pressure variable.

In describing the dimensionality of each term, the symbol theta (θ) is used to refer to angular displacement. Conventional thinking is to not consider it a dimension due to being defined as a ratio between two numbers. But since any linear displacement could be approximated as an angular displacement (think walking along the equator), it is helpful in analysis to elevate the phenomenon of angular displacement to the abstract levels of dimensional analysis.

F = ○□″ + ○′ □′ + ○″□

F = Force

○ = Mass	□ = Displacement
○′ = Friction	□′ = Velocity
○″ = Compliance	□″ = Acceleration

The prototype constitutive formula. A second-order ordinary linear differential equation as a model for energetic transfer. This example uses variables from the mechanical level.

Impedance, as the triple [○ , ○′ , ○″] can be represented as its inverse, admittance, with the inverse of resistance (the real component of impedance) as conductance, and the inverse of reactance (both inductive and capacitive) as susceptance.

$Z = X_L + R + X_C$ $\qquad$ $Y = B_L + G + B_C$

The complex terms (reactance) are sometimes combined and expressed as:

$Z = R + jX$

with:

$X_L = \omega L = 2\pi f L$ $\qquad$ $X_C = 1/\omega C = 1/2\pi f C$ $\qquad$ $X = X_L - X_C$

The use of the network model, with lumped impedance parameters has also found application in social and biomedical sciences as an effective model for understanding artificial and biological systems (Causal Analysis in Biomedicine and Epidemiology: Based on Minimal Sufficient Causation, by Mikel Aickin, CRC Press, 2002).

I've been working on a few extensions to this "Rosetta Stone." One extension is to add in gravity and light. As others have observed, this parallel is too obvious to ignore:

$$\bar{F} = -G\frac{m_1 m_2}{\bar{r}^2} \qquad \bar{F} = C\frac{q_1 q_2}{\bar{r}^2}$$

Perhaps the gravitational constant *G* can be viewed as a reactance property of earth.

Adding in light is another interesting exercise. Where does color temperature fit in? Perhaps refractance, absorptance, and reflectance are the three corresponding terms in "optical impedance" (or optical admittance). Absorption is a dissipation of energy, similar to resistance, reflection is analogous to spring compliance, and refraction analogous to inductance. Other optical quantities can be modeled, such as:

- Illuminance - luminous flux per unit area incident on a surface
- Luminance - luminous flux per unit solid angle and per unit projected area, in a given direction, at a point on a surface.
- Radiant flux - total power/energy of the incident radiation.

Another table of analogues is currently being worked out – a table of Analogy in Logical Action. This will identify parallels between physical impedance and logical impedance, and serve as a model for the informational domains, including data processing systems, cognitive models, belief systems, and the bioenergetics of emotional reactivity.

Analogy in Physical Action

Action Level	Pressure	Flow	Inductance	Resistance	1/Capacitance
Mechanical	Force	Velocity	Mass	Friction	Spring Constant
$F = Ma + Dv + Kx$	F	v	M	D	K
$[ML/T^2]=[M]*[L/T^2]+[M/T]*[L/T]+[M/T^2]*[L]$	$[ML/T^2]$	$[L/T]$	$[M]$	$[M/T]$	$[M/T^2]$
Rotational	Torque	Angular Velocity	Rotational Inertia	Rotational Damping	Torque Constant
$T = J\alpha + D_r\omega + K_r\theta$	T	ω	J	D_r	K_r
$[ML^2\theta/T^2]=[ML^2]*[\theta/T^2]+[ML^2/T]*[\theta/T]+[ML^2/T^2]*[\theta]$	$[ML^2\theta/T^2]$	$[\theta/T]$	$[ML^2]$	$[ML^2/T]$	$[ML^2/T^2]$
Hydraulic	Pressure	Flow	Density	Fluid Resistance	1/Fluid Capacitance
$p = \rho q' + R_f q + (1/C_f)A$	p	q	ρ	R_f	$1/C_f$
$[M/LT^2]=[M/L^3]*[L^2/T^2]+[M/L^3T]*[L^2/T]+[M/L^3T^2]*[L^2]$	$[M/LT^2]$	$[L^2/T]$	$[M/L^3]$	$[M/L^3T]$	$[M/L^3T^2]$
Acoustic	Acoustic Pressure	Diaphragm Velocity	Acoustic Inertance	Acoustic Resistance	Stiffness
$p = I\xi'' + R_a\xi' + s\xi$	p	ξ'	I	R_a	s
$[M/L^3T^2]=[M/L^4]*[L/T^2]+[M/L^4T]*[L/T]+[M/L^4T^2]*[L]$	$[M/L^3T^2]$	$[L/T]$	$[M/L^4]$	$[M/L^4T]$	$[M/L^4T^2]$

Action Level	**Pressure**	**Flow**	**Inductance**	**Resistance**	**1/Capacitance**
Electrical	Voltage	Current	Inductance	Resistance	1/Capacitance
$E = Li' + Ri + (1/C)q$	E	i	L	R	$1/C$
$[ML^2/QT^2]=[ML^2/Q^2]*[Q/T^2]+[ML^2/Q^2T]*[Q/T]+[ML^2/Q^2T^2]*[Q]$	$[ML^2/QT^2]$	$[Q/T]$	$[ML^2/Q^2]$	$[ML^2/Q^2T]$	$[ML^2/Q^2T^2]$
Magnetic	Magneto-motive Force	Flux	Magnetic Inductance	Reluctance	1/Magnetic Capacitance
$\mathfrak{F} = \mathfrak{J}\varphi' + \mathfrak{R}\varphi + (1/\mathfrak{C})q$	$\mathfrak{F}$	φ	$\mathfrak{J}$	$\mathfrak{R}$	$1/\mathfrak{C}$
$[Q/T]=[Q^2T/ML^2]*[ML^2/QT^2]+[Q^2/ML^2]*[ML^2/QT]+[Q^2/ML^2/Q]*[ML^2/Q]$	$[Q/T]$	$[ML^2/QT]$	$[Q^2T/ML^2]$	$[Q^2/ML^2]$	$[Q^2/ML^2T]$
Solid-Mechanical	Stress	Strain	Density	Rigidity	Modulus of Elasticity
$\tau = \rho\varepsilon'' + \mu_p\varepsilon' + E\varepsilon$	τ	ε'	ρ	μ_p	E
$[M/LT^2]=[M/L]*[1/T^2]+[M/LT]*[1/T]+[M/LT^2]*[1]$	$[M/LT^2]$	$[1/T]$	$[M/L]$	$[M/LT]$	$[M/LT^2]$
Thermal	Temperature Difference	Heat Flow	Thermal Inductance	Thermal Resistance	1/ThermalCapacitance
$\tau = I_T q_T' + R_T q_T + (1/C_p) w_T$	τ	q_T	I_T	R_T	$1/C_p$
$[ML^2/T^2]=[ML^2/H]*[H/T^2]+[ML^2/HT]*[H/T]+[ML^2/HT^2]*[H]$	$[ML^2/T^2] = [K]$	$[H/T]$	$[ML^2/H]$	$[ML^2/HT]$	$[ML^2/HT^2]$